On rigorous verification of the crossed mapping condition

Computation and Dynamics @ ICERM

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荒井迅

Zin ARAI

Hokkaido University

joint work with

Yutaka Ishii (Kyushu University)

Background and Main Results

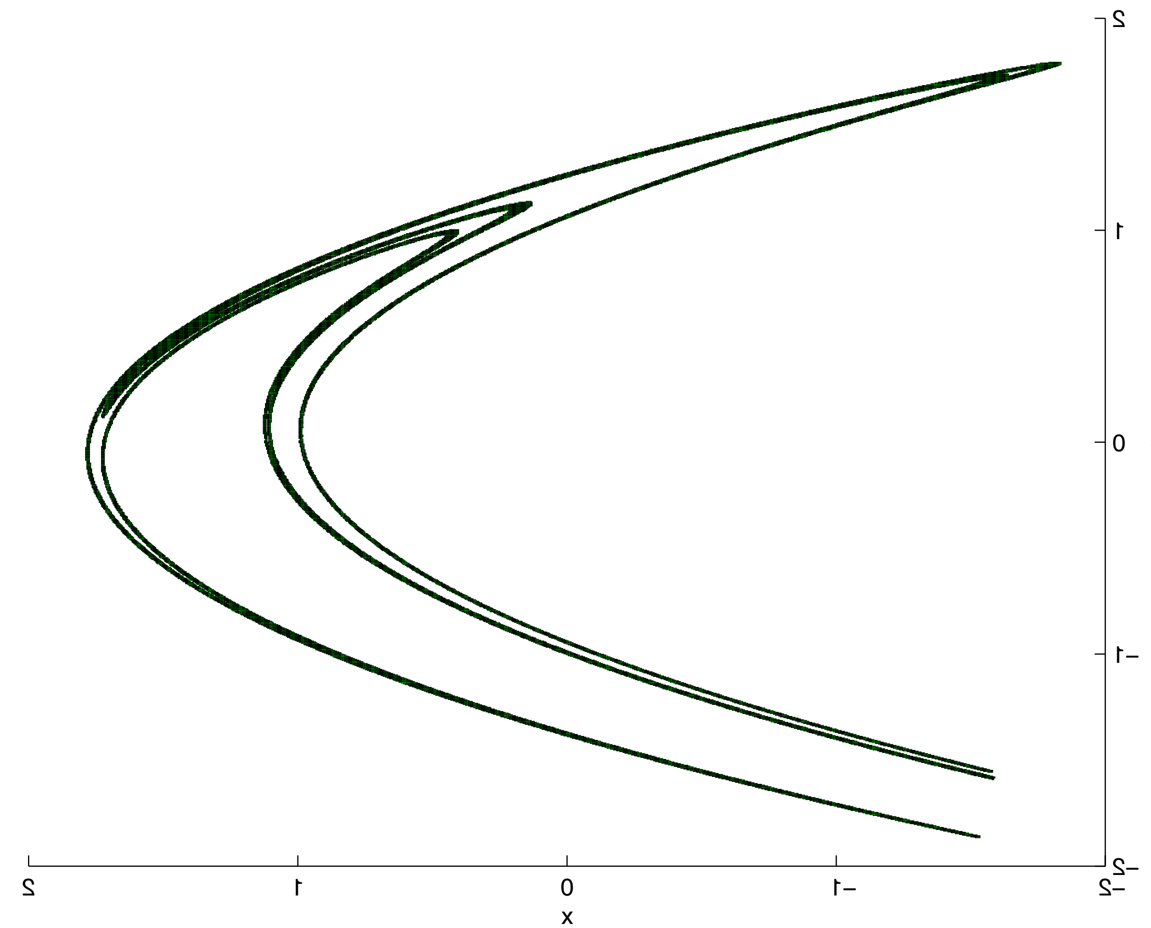
The Hénon Map

The *Hénon family* on $\mathbb{R}^2$:

$$f_{a,b} : \mathbb{R}^2 \ni (x, y) \longmapsto (x^2 - a - by, x) \in \mathbb{R}^2,$$

where $(a, b) \in \mathbb{R} \times \mathbb{R}^{\times}$.

Michel Hénon
1931-2013

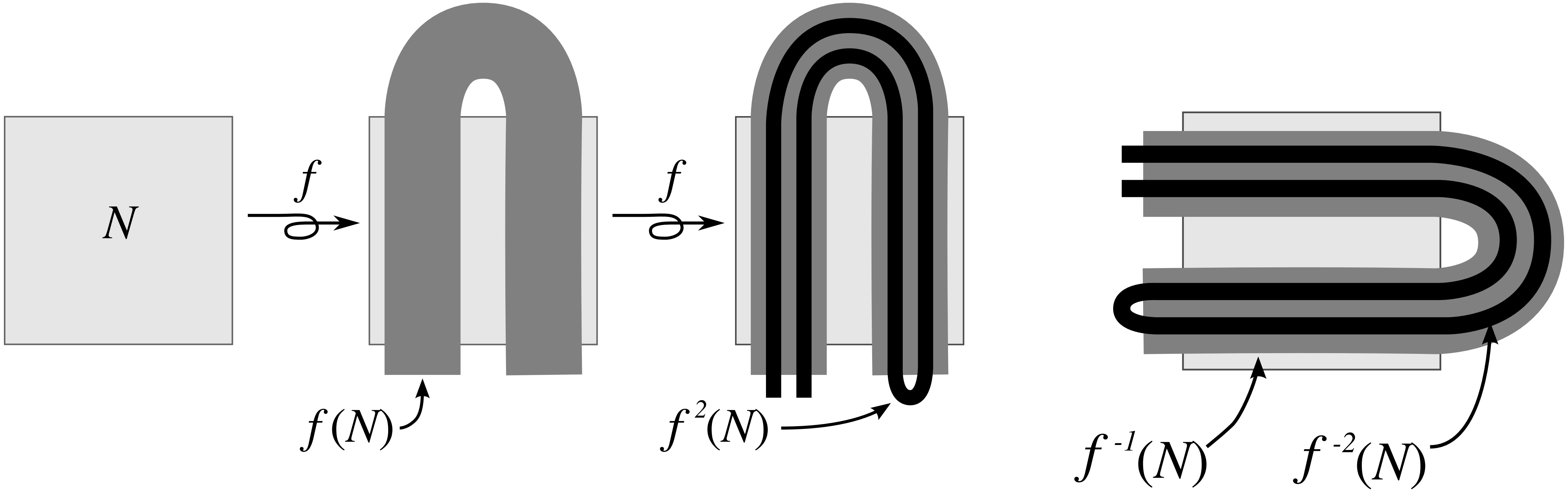


strange attractor for $(a, b) = (1.4, 0.3)$

Smale Horseshoe

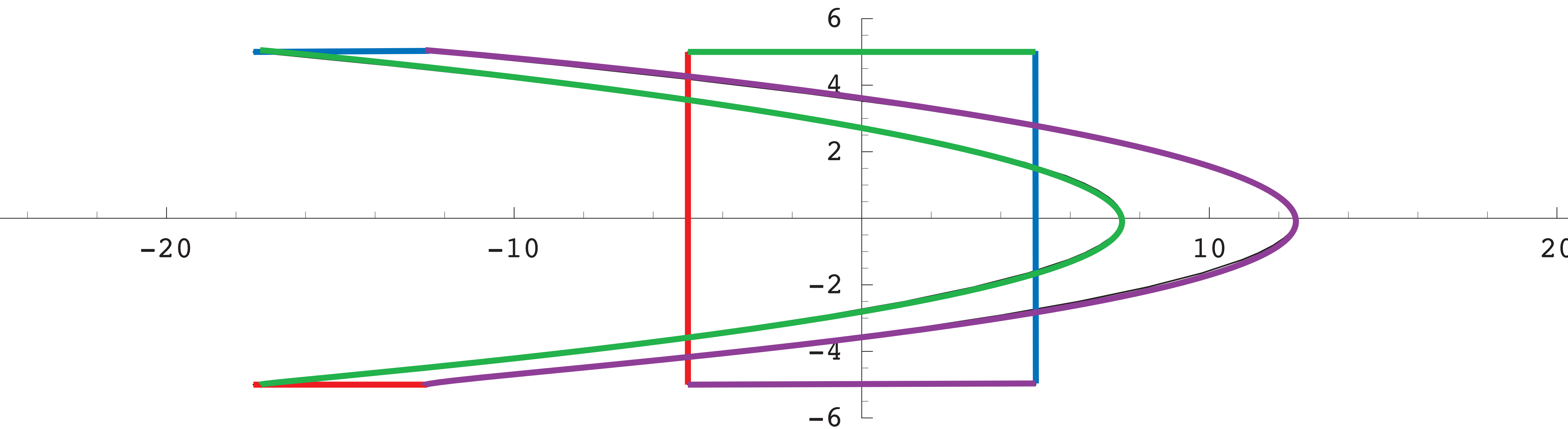
Stephan Smale

“Finding a horseshoe on the beaches of Rio”

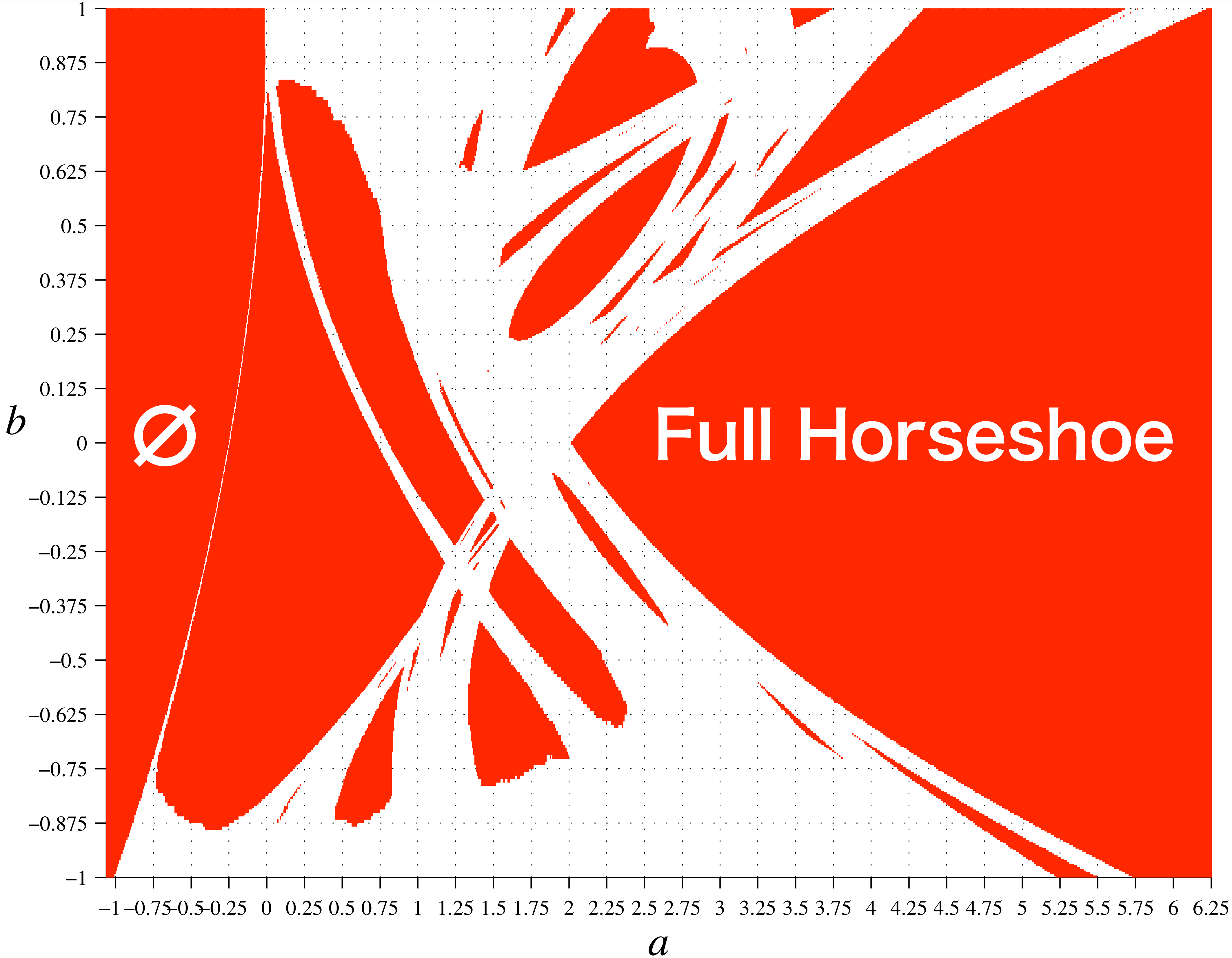


Horseshoe in the Hénon family

[Devaney and Nitecki 1979] For any fixed b, if a is sufficiently large, then the non-wandering set of the Hénon map is uniformly hyperbolic full horseshoe.

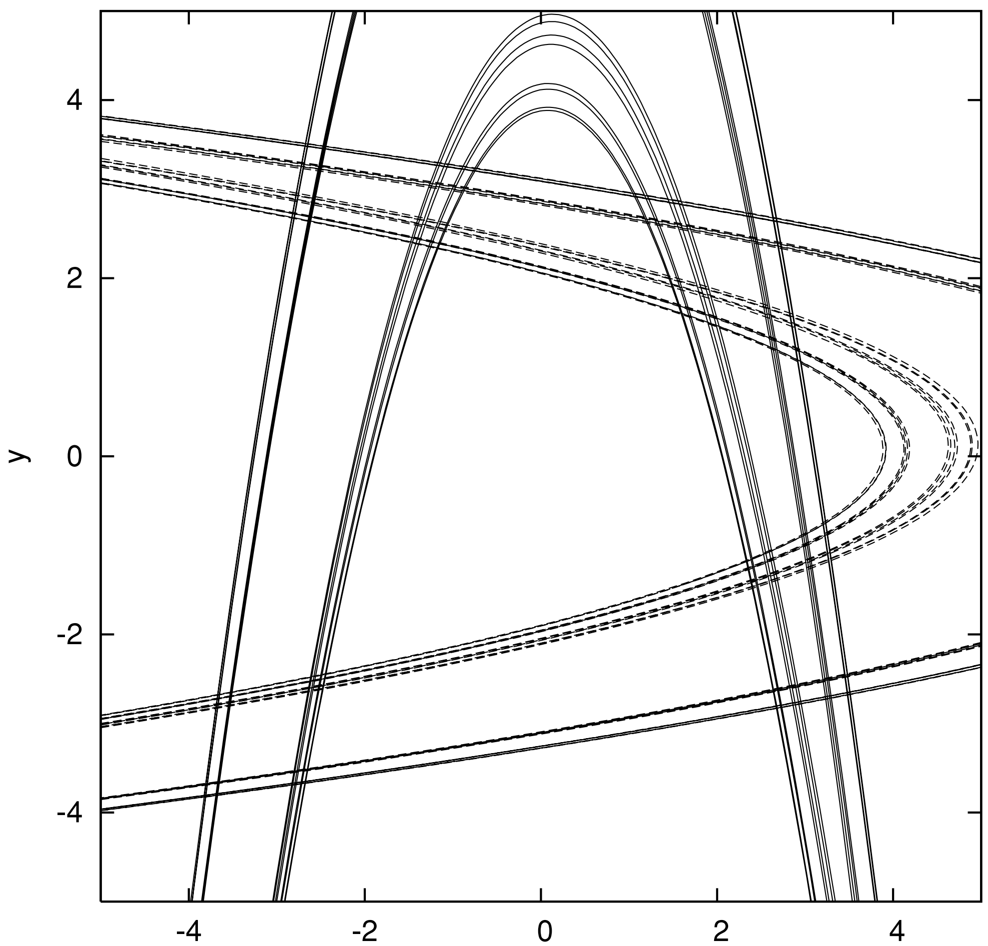


Hyperbolic Regions

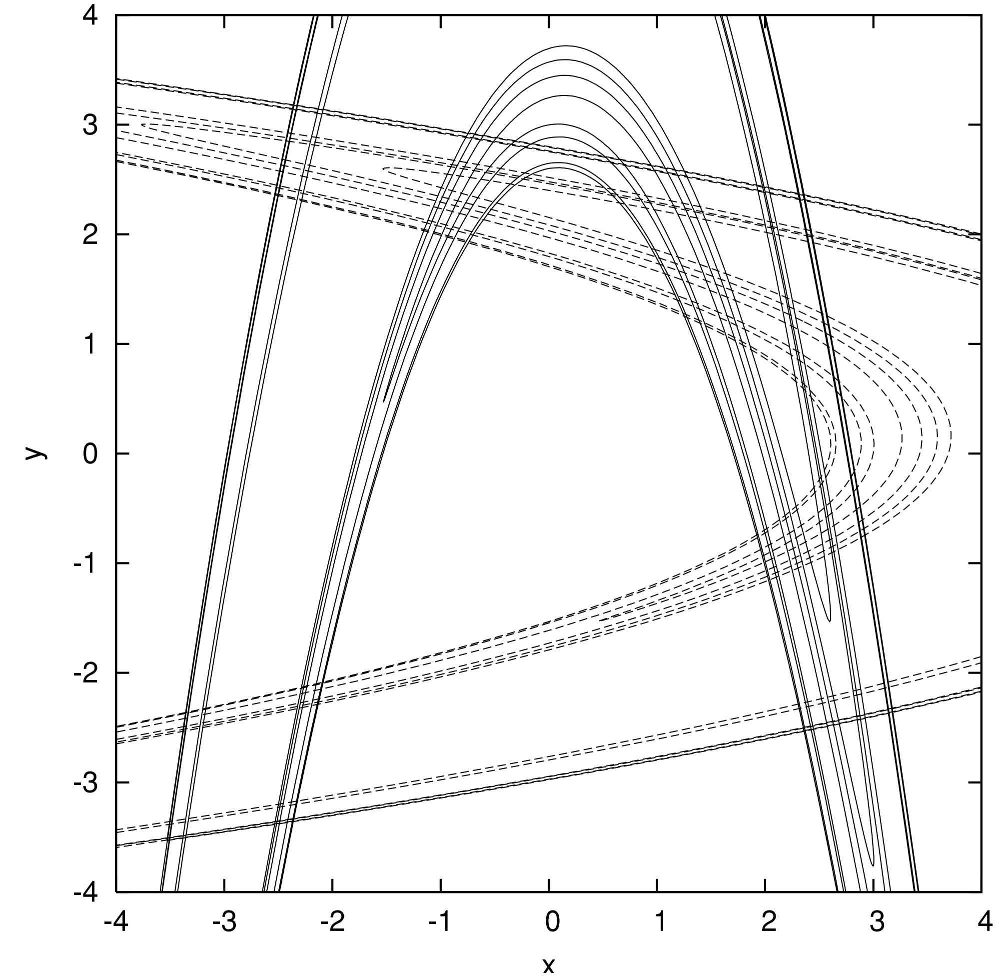


Hyperbolic parameter region of the real Hénon map (ZA, Experimental Math. 16 2007)

Tangency



a = 6.0, b = 1.0



a = 5.4, b = 1.0

It is likely that the "first" bifurcation is a tangency between unstable and stable manifolds of the fixed pt.

First Bifurcation Problem

Problem: Determine the type of “first” bifurcation and describe the dynamics at the bifurcation

Known Results:

Real Dynamics: Cao-Luzzatto-Rios (2008)

Complex Dynamics: Bedford-Smillie (2006) (for small $|b|$, $|b| < 0.06$)

ZA-Ishii (2015) (for any b)

The Main Result

$$\mathcal{M}^{\times} \equiv \{(a, b) \in \mathbb{R}\times\mathbb{R}^{\times} : f_{a,b} \text{ attains the maximal entropy } \log 2\}$$

$$\mathcal{H}^{\times} \equiv \{(a, b) \in \mathbb{R} \times \mathbb{R}^{\times} : f_{a,b} \text{ is a hyperbolic horseshoe on } \mathbb{R}^2\}.$$

Main Theorem (ZA & Y. Ishii 2014)

There exists an analytic function $a_{\mathrm{tgc}} : \mathbb{R}^{\times} \to \mathbb{R}$ from the b-axis to the a-axis in the parameter space of $f_{a,b}$ with $\lim_{b\to 0} a_{\mathrm{tgc}}(b) = 2$ s.t.

- $a > a_{\mathrm{tgc}}(b)$ *iff* $(a, b) \in \mathcal{H}^{\times}$,
- $a \geq a_{\mathrm{tgc}}(b)$ *iff* $(a, b) \in \mathcal{M}^{\times}$.

Moreover, when $a = a_{\mathrm{tgc}}(b)$, the map $f_{a,b}$ has exactly one orbit of tangencies of the stable/unstable manifolds of certain fixed points.

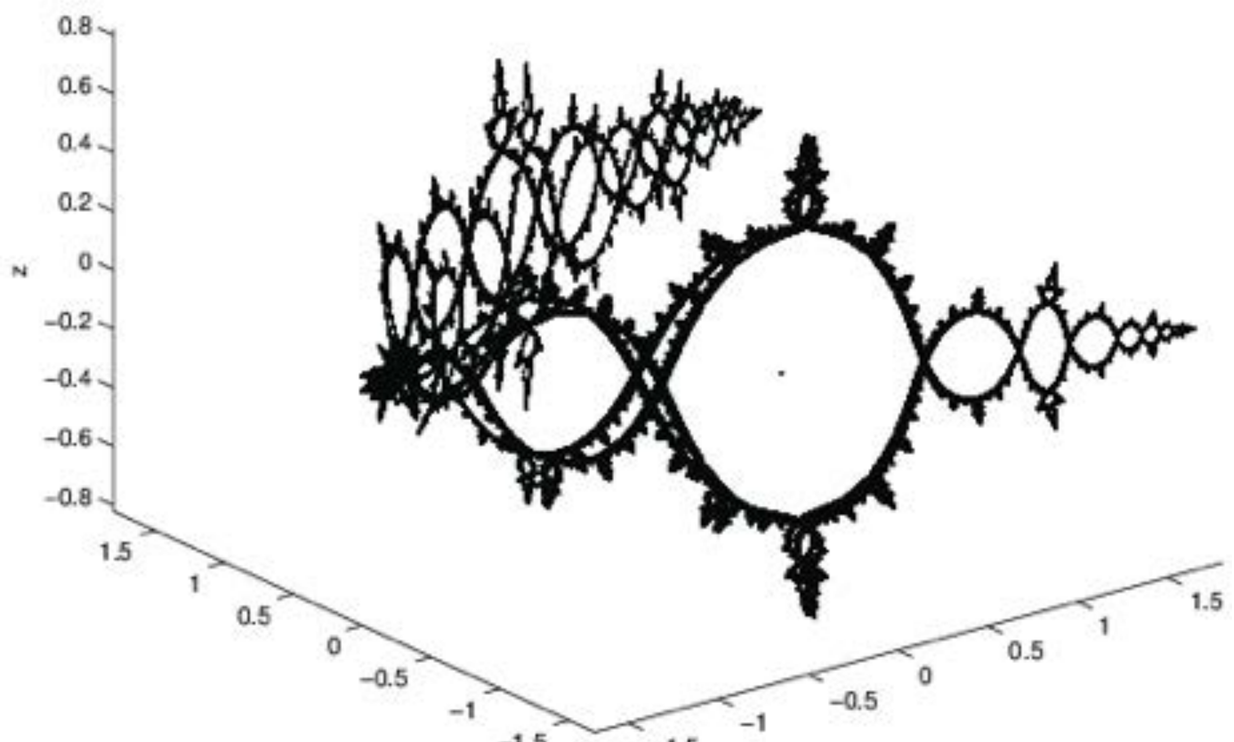
Previously shown by Bedford and Smillie (2006) for $|b| < 0.06$.

Our Strategy

Basically, we fallow the proof of Bedford and Smillie: Extend both the dynamical and parameter planes from $\mathbb{R}^2$ to $\mathbb{C}^2$ and investigate the complexified dynamics.

The dynamics is, however, much more complicated than Bedford-Smillie's near-1-dim case.

Therefore, we introduce "projective box" methods and also use rigorous numerics.



The projection of the filled Julia set of the Hénon map with $a = 1.1875$, $b = 0.15$ to $\mathbb{R}^3 =<\mathrm{Re}\,x, \mathrm{Re}\,y, \mathrm{Im}\,x>$

A Partition of the Parameter Space

Quasi-Trichotomy

We classify any Hénon map $f_{a,b}$ into three types (not exclusive).

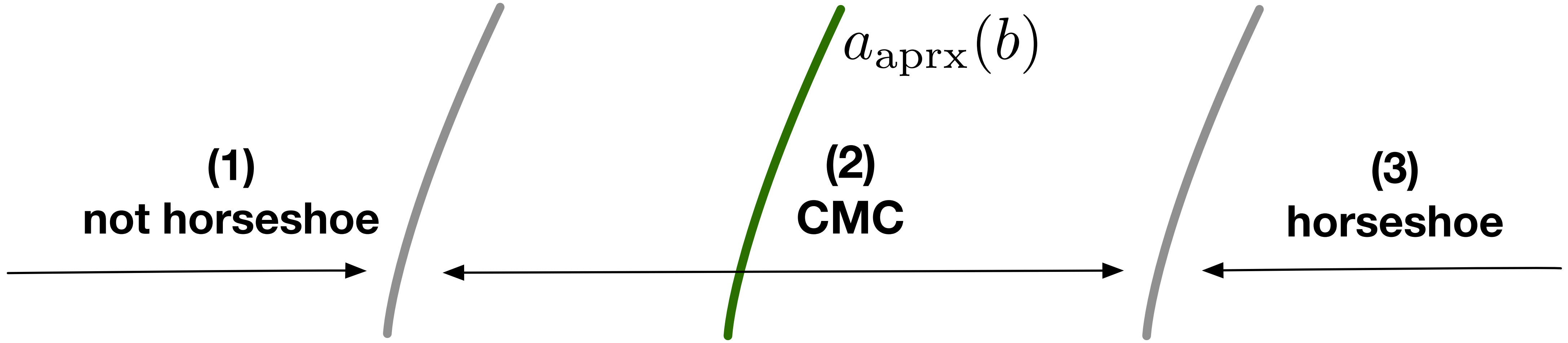
Theorem (Quasi-Trichotomy)

One can construct a piecewise affine function $a_{\mathrm{aprx}} : \mathbb{R}^\times \to \mathbb{R}$ (whose graph approximates $\partial\mathcal{M}$ and $\partial\mathcal{H}$) so that

(1) for $(a, b) \in \mathbb{R} \times \mathbb{R}^\times$ with $a \leq a_{\mathrm{aprx}}(b) - 0.1$, the Hénon map $f_{a,b}$ satisfies $h_{\mathrm{top}}(f_{a,b}|_{\mathbb{R}^2}) < \log 2$,

(2) for $(a, b) \in \mathbb{C} \times \mathbb{R}^\times$ with $|a - a_{\mathrm{aprx}}(b)| \leq 0.1$, the Hénon map $f_{a,b}$ satisfies the (CMC) for a family of boxes $\{\mathcal{B}_i\}_i$ in $\mathbb{C}^2$,

(3) for $(a, b) \in \mathbb{R} \times \mathbb{R}^\times$ with $a \geq a_{\mathrm{aprx}}(b) + 0.1$, the Hénon map $f_{a,b}$ is a hyperbolic horseshoe on $\mathbb{R}^2$.



Proof of Quasi-trichotomy (1)

Use a result of Bedford-Lyubich-Smillie (1993):

$$h_{\mathrm{top}}(f_{a,b}|_{\mathbb{R}^2}) = \log 2 \quad \text{if and only if} \quad K_{f_{a,b}} \subset \mathbb{R}^2$$

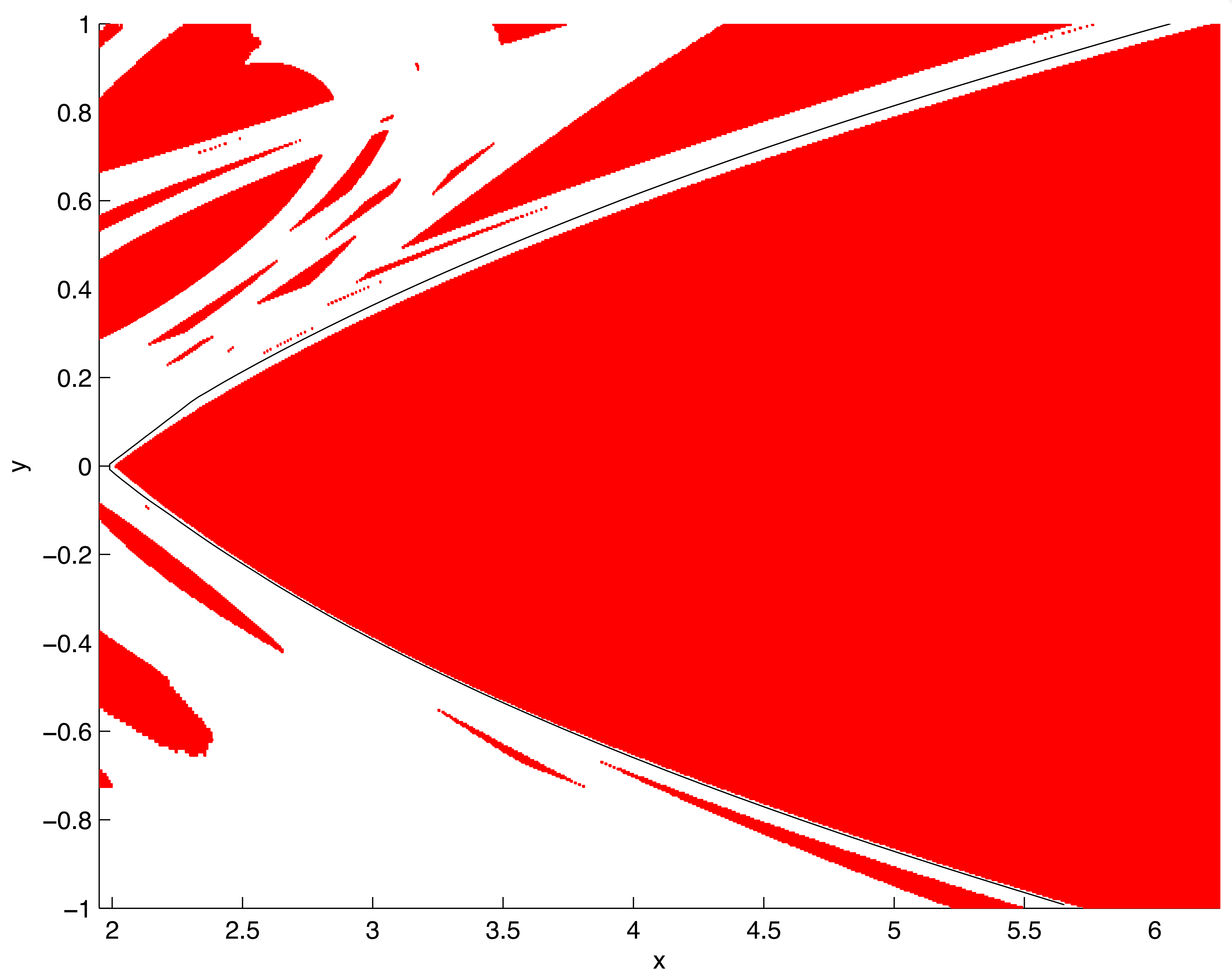
for a real Hénon map $f_{a,b}$, where K_f is the set of points in $\mathbb{C}^2$ whose forward and backward orbits by f are both bounded.

In particular, it is enough to find a saddle periodic point in $\mathbb{C}^2 \setminus \mathbb{R}^2$.

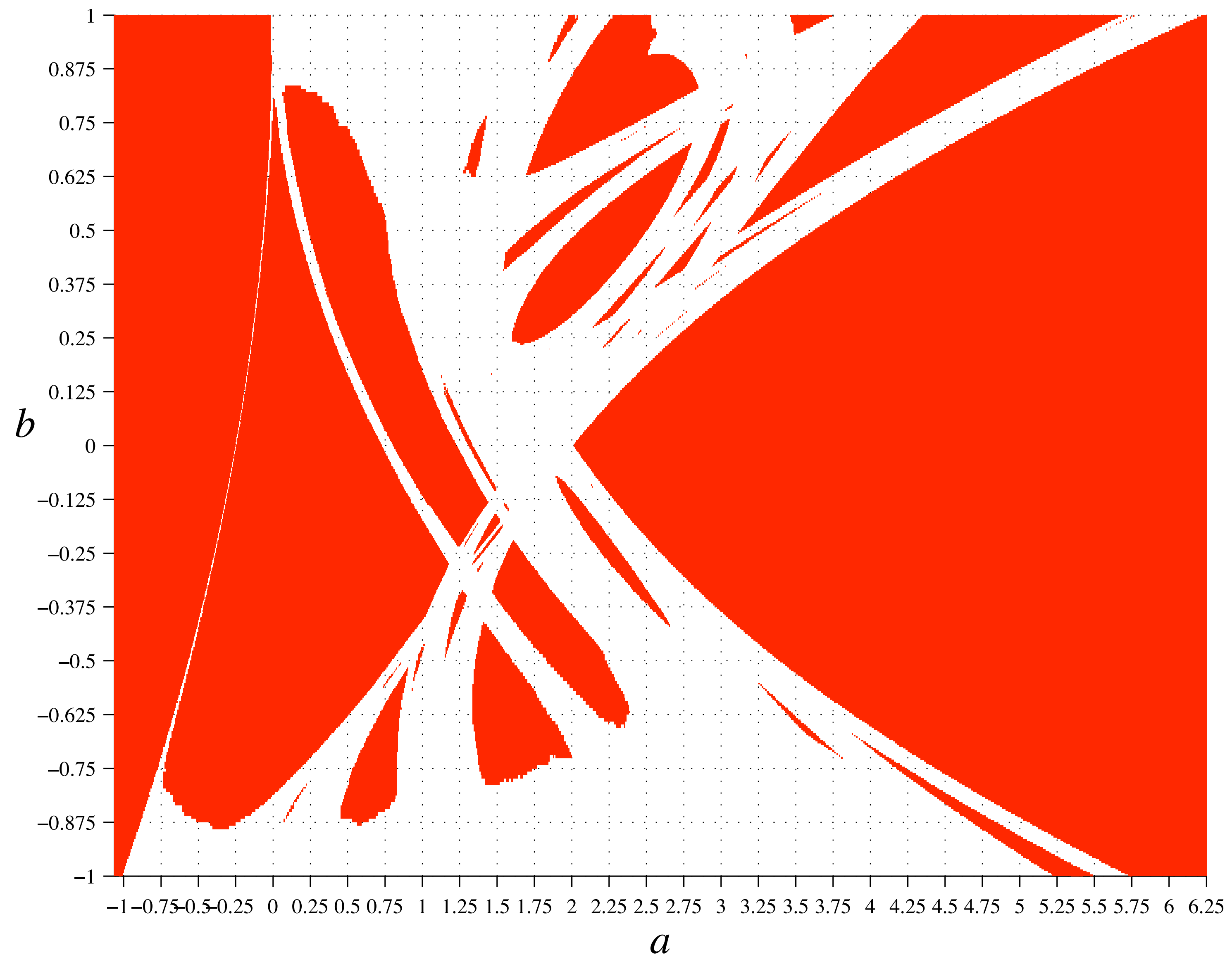
Rigorous Numerics

Using the Interval Krawczyk Method, we can prove the existence of such saddle point (of period 7) for all parameter values satisfying $a \leq a_{\mathrm{aprx}}(b) - 0.1$.

SN Curves of Period 7



Proof of Quasi-trichotomy (3)



Rigorous Numerics

The uniform hyperbolicity for all parameter values satisfying $a \geq a_{\mathrm{aprx}}(b) + 0.1$ follows from my previous work (ZA, Experimental Math 16, 2007).

The remaining part, (2)

On the remaining parameter region (2), by using the crossed mapping condition, we claim that we can control

(b > 0) the homoclinic tangency associated to the fixed point on the 1st quadrant

(b < 0) the heteroclinic tangency between the two fixed points,

and these are the first bifurcations.

Remark: Thanks to the Bedford-Lyubich-Smillie theorem, we don't need to consider tangencies associated to other periodic points.

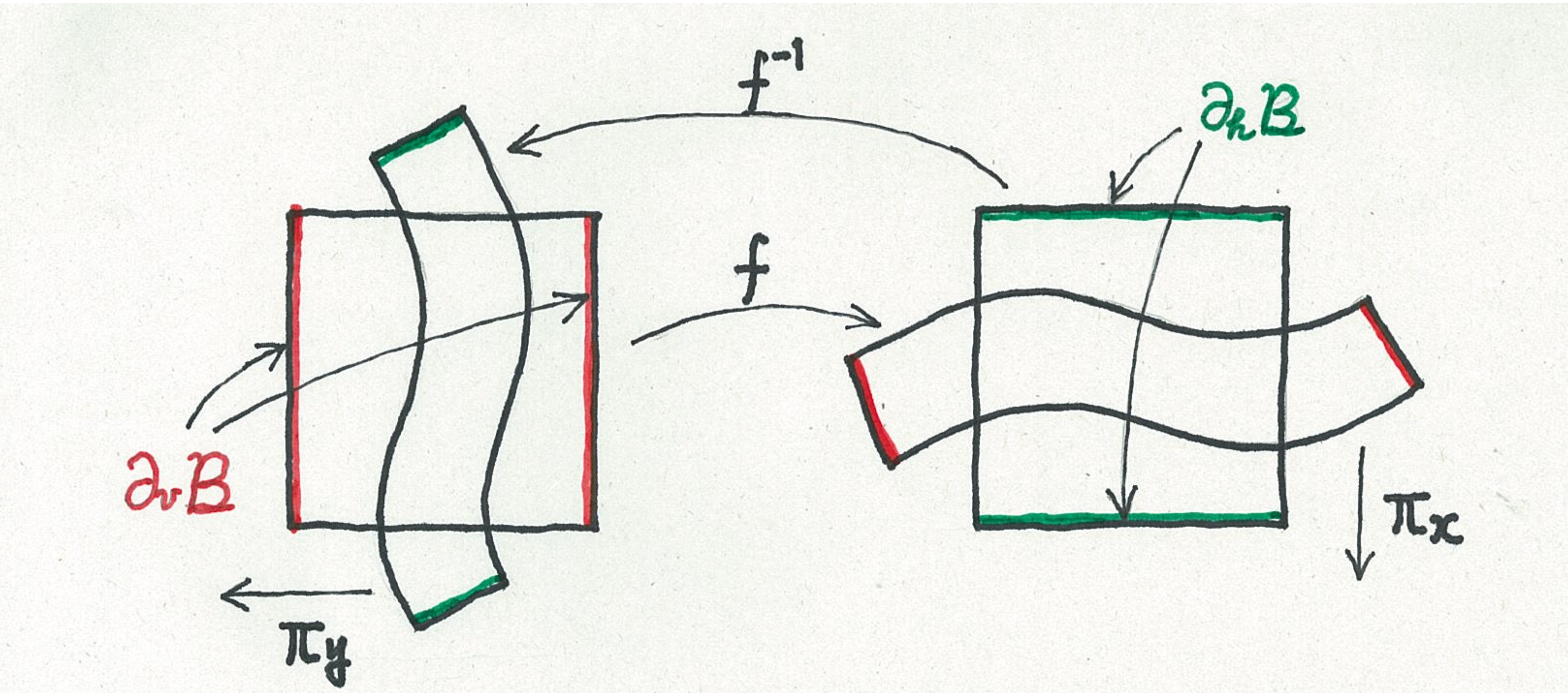
Crossed Mapping Condition

A *box* is a product set $\mathcal{B} \equiv U_x \times U_y \subset \mathbb{C}^2$, where U_x and U_y are domains in $\mathbb{C}$. Set $\partial_v \mathcal{B} \equiv \partial U_x \times U_y$ and $\partial_h \mathcal{B} \equiv U_x \times \partial U_y$.

Let $f = f_{a,b} : \mathbb{C}^2 \to \mathbb{C}^2$ be a Hénon map over $\mathbb{C}$.

Definition (J. H. Hubbard & R. W. Oberste-Vorth)

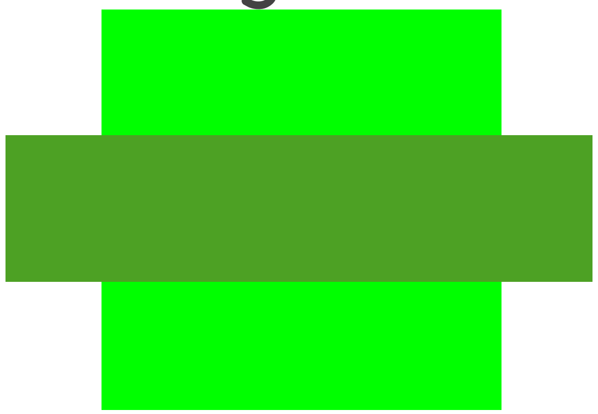
We say $f : \mathcal{B} \to \mathcal{B}$ satisfies the *crossed mapping condition (CMC)* if $\pi_x \circ f(\partial_v \mathcal{B}) \cap U_x = \emptyset$ and $\pi_y \circ f^{-1}(\partial_h \mathcal{B}) \cap U_y = \emptyset$ hold.



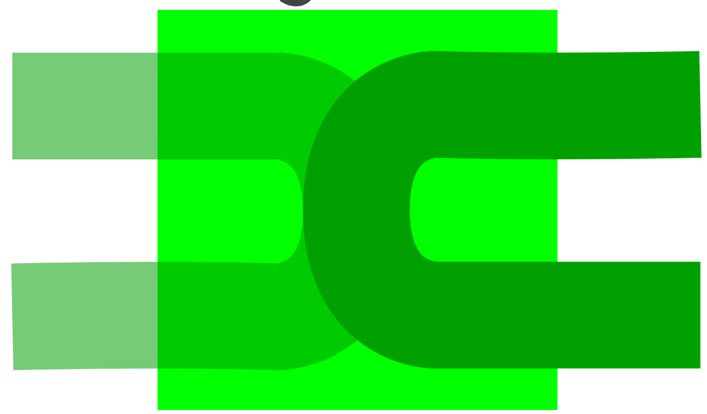
Degree and intersections

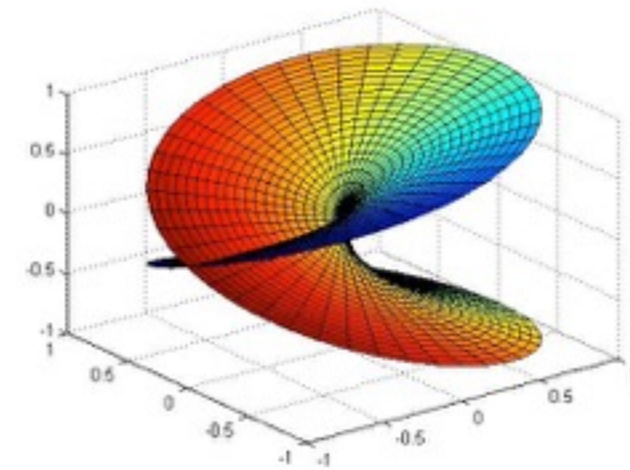
- We can define the degree of a crossed mapping

degree 1



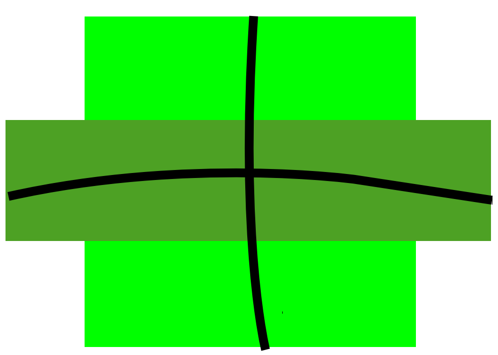
degree 2



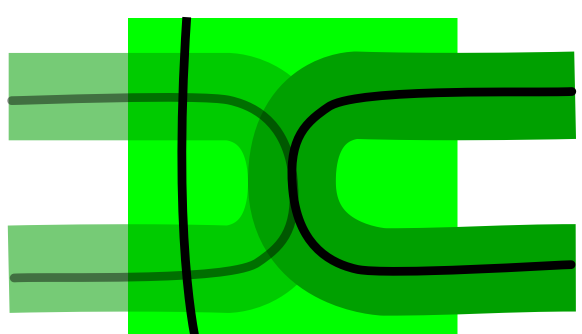


- If $f : \mathcal{B}_1 \to \mathcal{B}_2$ is a crossed mapping and D_h is a "horizontal disc" in $\mathcal{B}_1$, D_v a "vertical disc" in $\mathcal{B}_2$,

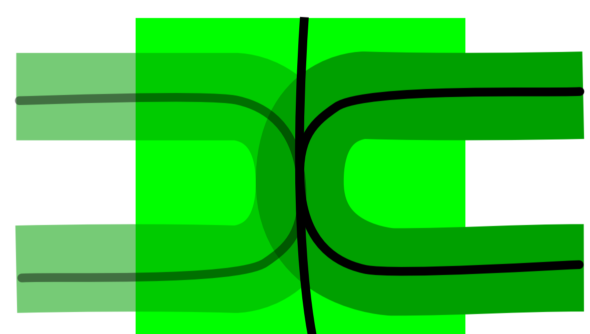
$\#(D_v \cap f(D_h)) = \text{degree of } f$ (counted with multiplicity)



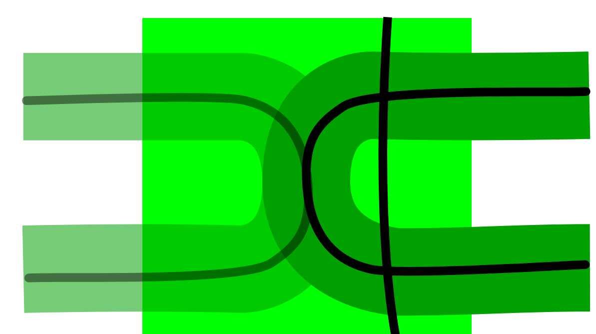
degree 1



2 imaginary intersections



tangency (multiplicity 2)



2 real intersections

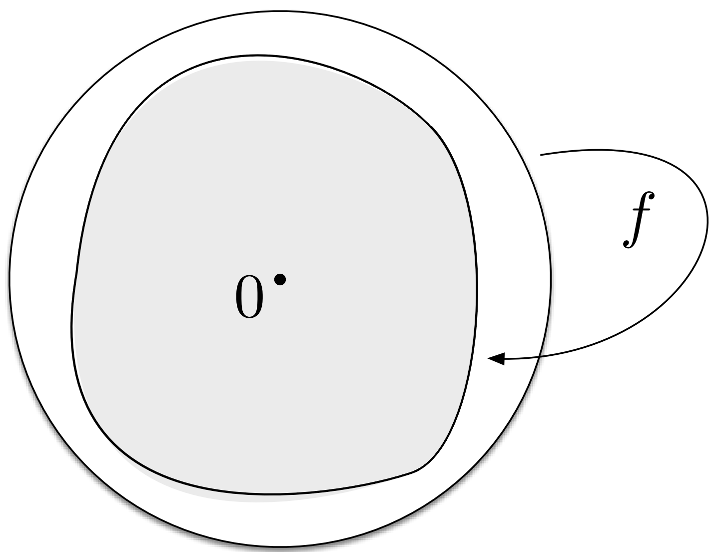
Topological Hyperbolicity

Schwarz Lemma

Let $\mathbb{D} \subset \mathbb{C}$ be the unit open disk and $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map with $f(0) = 0$.

Then,

- f is a rotation about the origin, or,
- $|f'(0)| < 1$ and $f^n(z) \to 0$ for all $z \in \mathbb{D}$.



In other words,

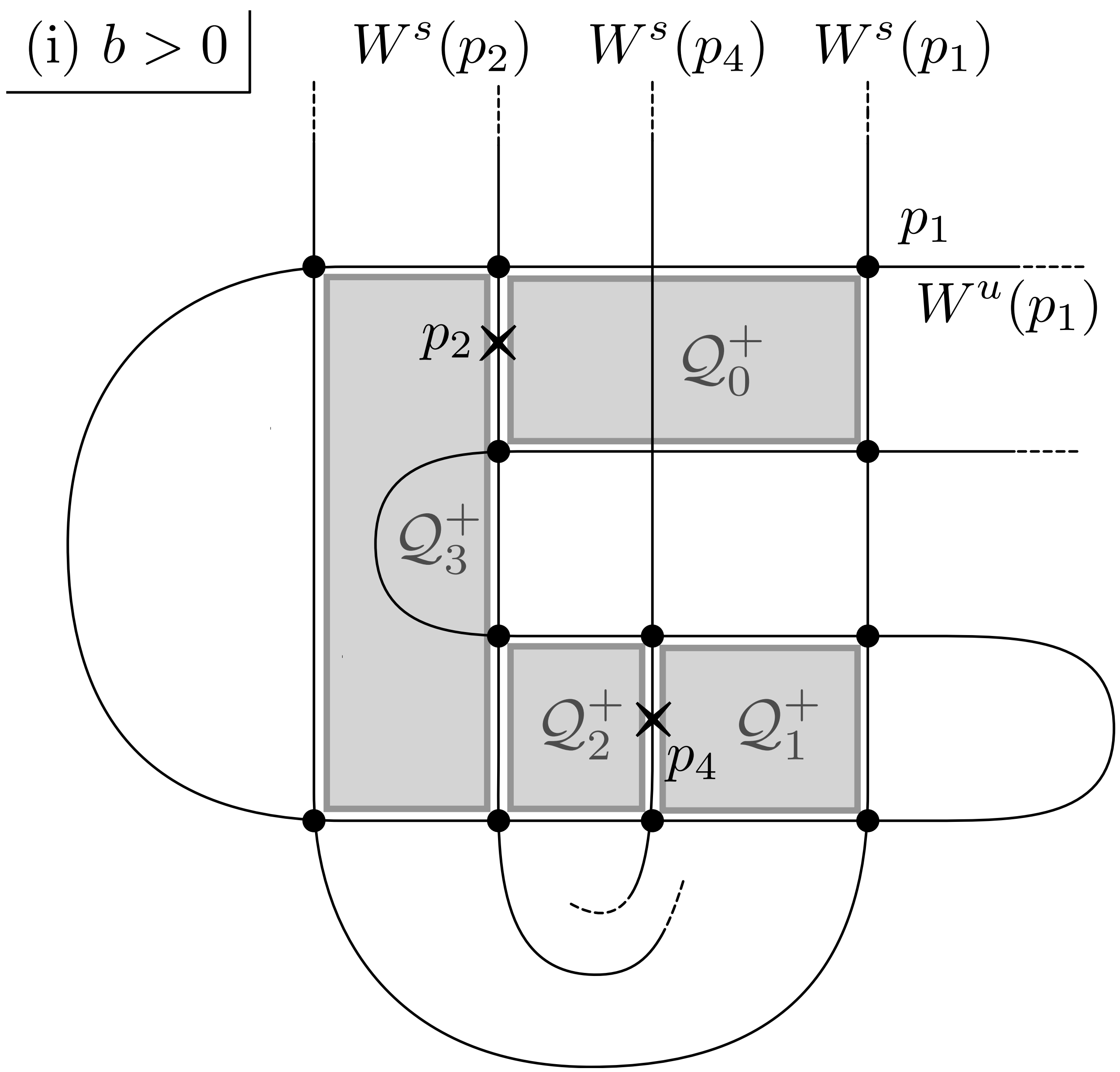
topological hyperbolicity $+$ holomorphic rigidity

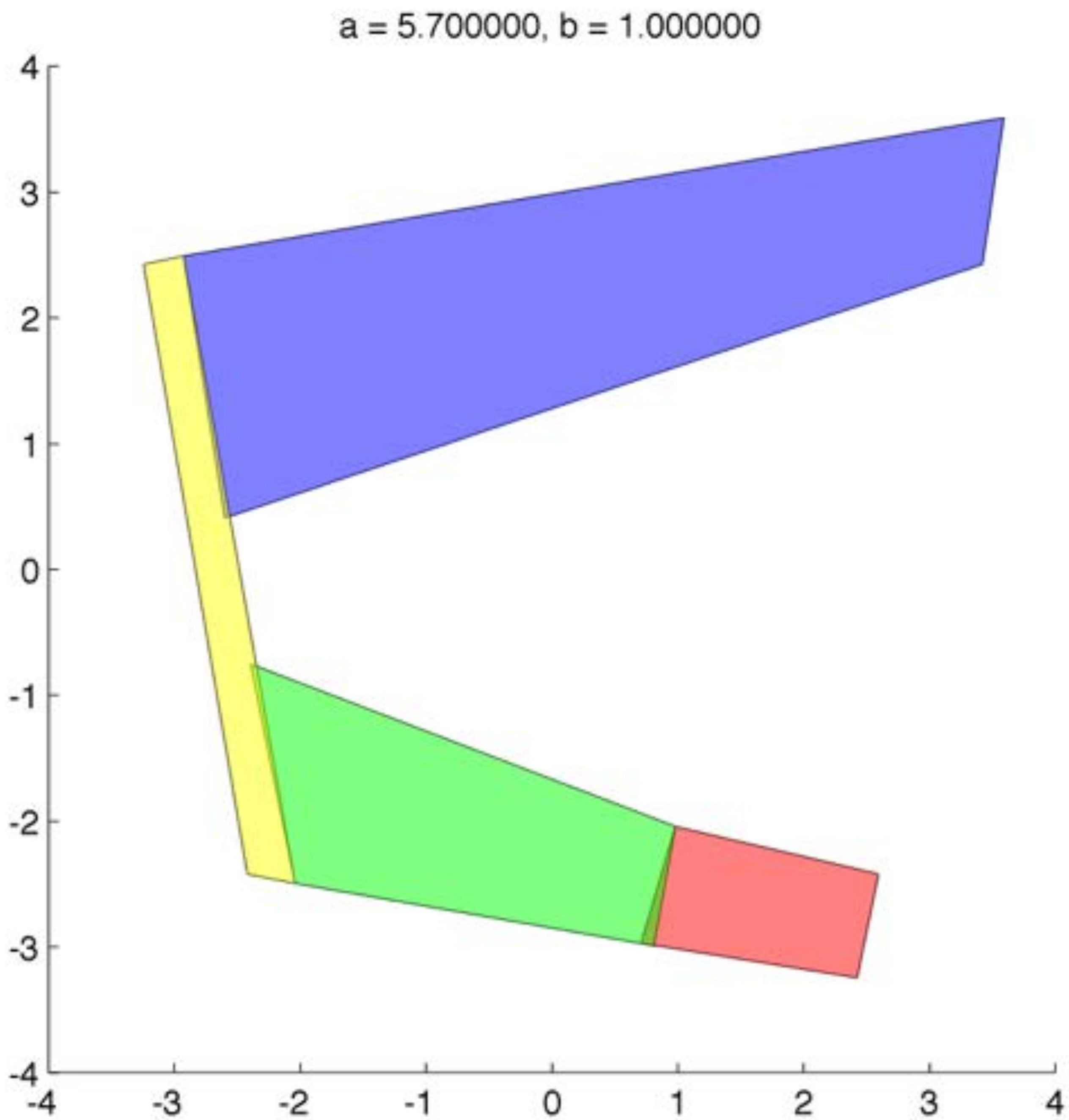
$\Rightarrow$ hyperbolicity

Real Boxes

Real Boxes

We define "real boxes" according to the trellis of the map.

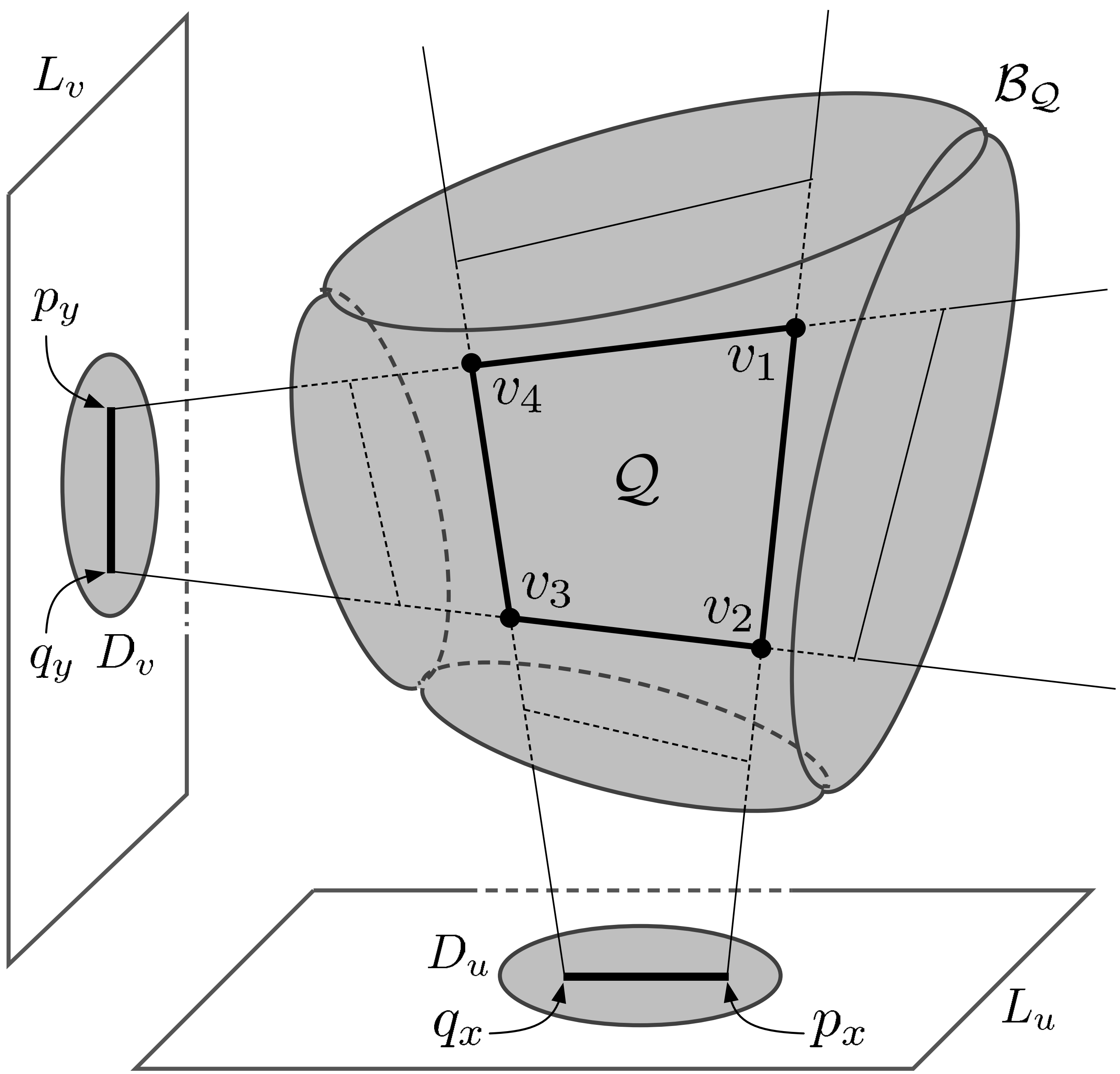




Complex Boxes

Complex Boxes

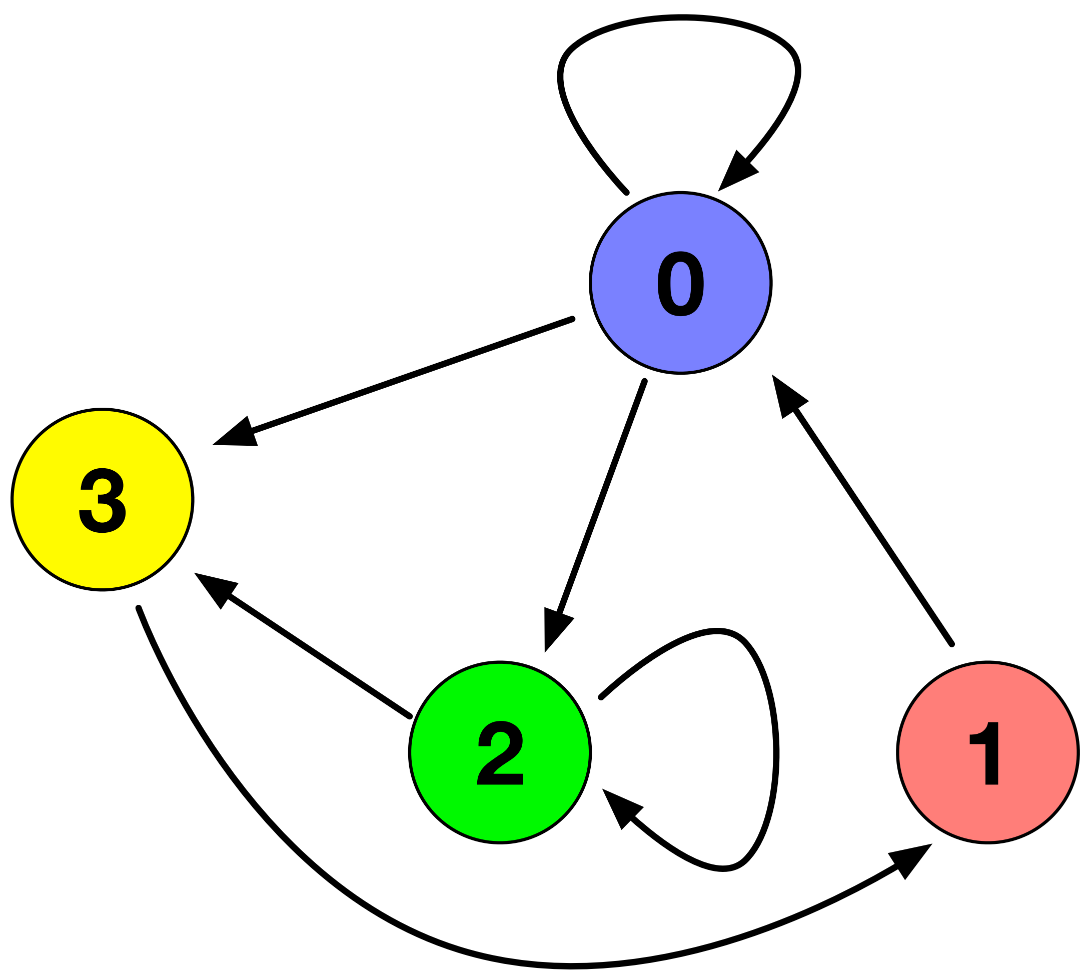
Real boxes are enlarged to complex boxes using projective coordinates.

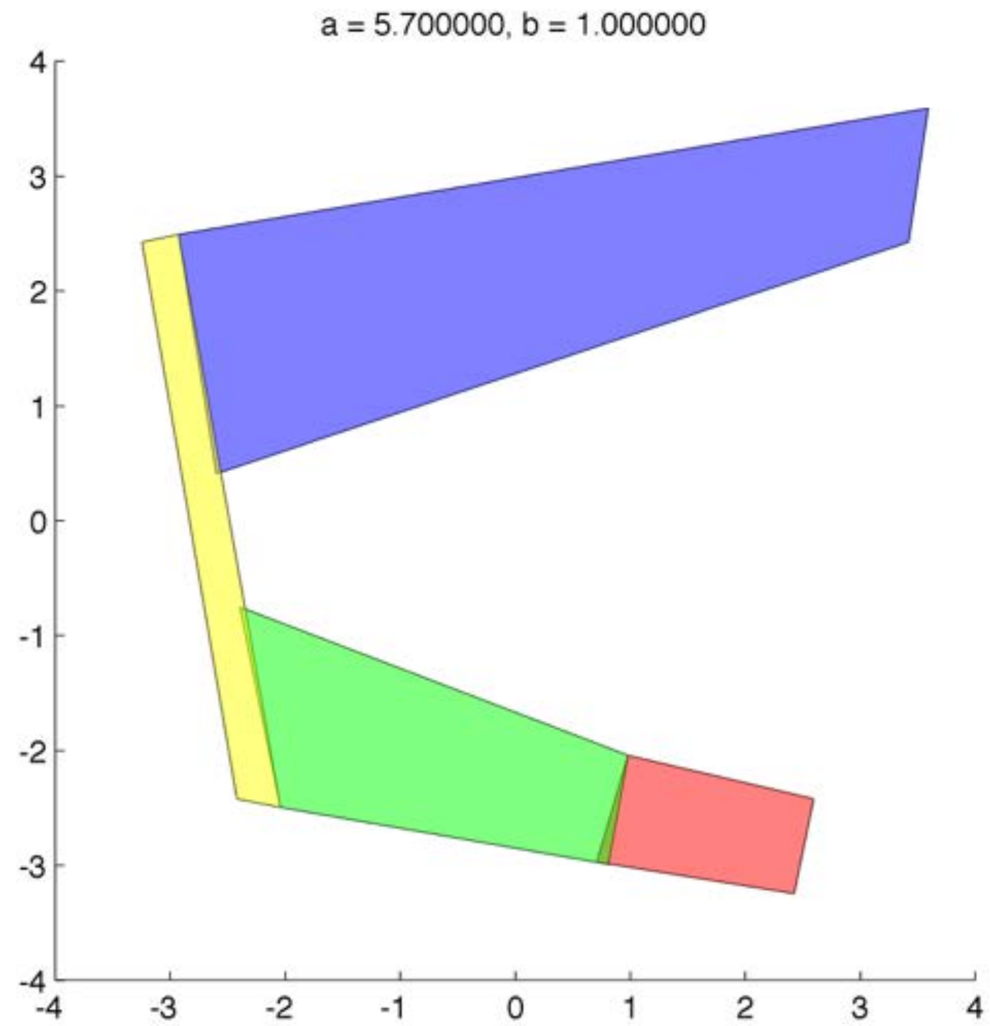


Rigorous Numerics

Rigorous Numerics

For all parameter values $|a - a_{\text{aprx}}(b)| \leq 0.1$ and all "admissible" pairs (i, j), we check that $f_{a,b} : B_i \to B_j$ satisfies CMC.

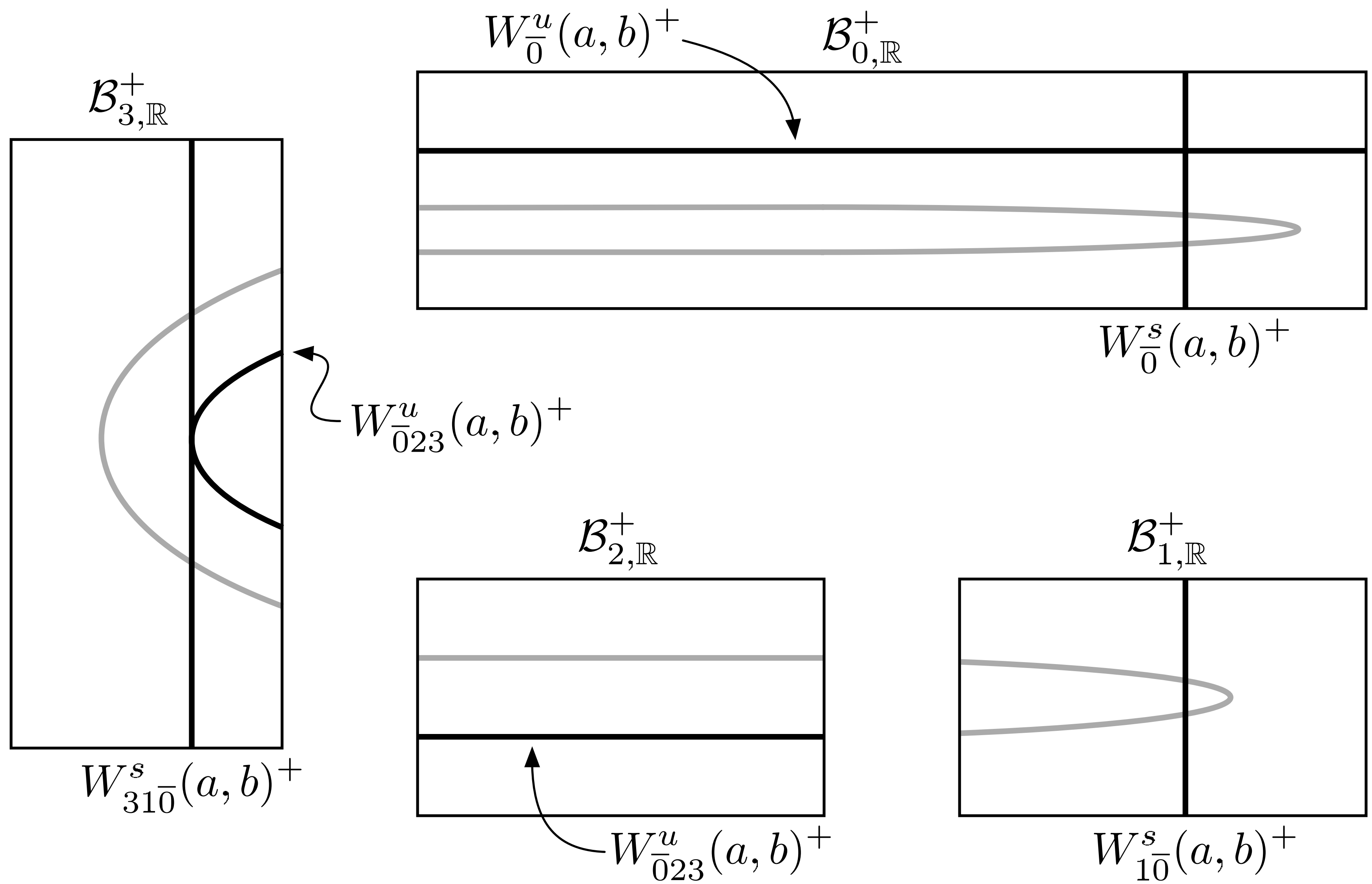




Locate the Tangency

Symbolic Decomposition

Family of boxes $\{\mathcal{B}_i\}_i$ enables us to define a symbolic encoding of pieces of the invariant manifolds $V^{u/s}(p)$ in $\mathbb{C}^2$, but with overlaps.



The (CMC) in the quasi-trichotomy assures that these pieces are "nice" holomorphic disks in $\mathcal{B}_i$.

Maximal Entropy Criterion

Now we return to the real dynamics $f_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$. Denote by $W_*^{u/s}(a,b)$ the real part of $V_*^{u/s}(a,b)$. We have the following characterization of $\mathcal{M}^\times$ in terms of "special pieces".

Theorem (Maximal Entropy)

Suppose that $(a,b) \in \mathcal{F}_{\mathbb{R}} = \mathcal{F} \cap \mathbb{R}^2$.

- *When* $b > 0$, *we have*
$$\mathrm{Card}(W^s_{31\bar{0}}(a,b) \cap W^u_{\bar{0}23}(a,b)) \geq 1 \iff (a,b) \in \mathcal{M}^\times.$$
- *When* $b < 0$, *we have*
$$\mathrm{Card}(W^s_{41\bar{0}}(a,b) \cap W^u_{\bar{4}34124}(a,b)_{\mathrm{inner}}) \geq 1 \iff (a,b) \in \mathcal{M}^\times.$$

Proof of Theorem (Maximal Entropy) uses the following result of Bedford-Lyubich-Smillie (1993). For a real Hénon map f, TFAE.

- $h_{\mathrm{top}}(f|_{\mathbb{R}^2}) = \log 2$,
- if p_1 and p_2 are saddles of f, then $V^u(p_1) \cap V^s(p_2) \subset \mathbb{R}^2$.

Special Pieces

Proposition (Special Pieces)

Assume the LHS of Theorem (Maximal Entropy). Then,

- $W^s_{31\overline{0}}(a,b)$ *is the left-most piece among* $D_3 \cap W^s(p)$ *and* $W^u_{\overline{0}23}(a,b)$ *is the inner-most piece among* $D_3 \cap W^u(p)$ *for* $b > 0$.
- $W^s_{41\overline{0}}(a,b)$ *is the left-most piece among* $D_4 \cap W^s(p)$ *and* $W^u_{\overline{434124}}(a,b)_{\mathrm{inner}}$ *is the inner-most piece among* $D_4 \cap W^u(q)$ *for* $b < 0$.

This excludes the possibility of other tangencies before the first tangency of "special pieces".

Complex Tangency Loci

The previous theorems indicates that the tangency between those pieces of invariant manifolds are responsible for the "first" bifurcation.

Definition

We define

$$\mathcal{T}^+ \equiv \{(a,b) \in \mathbb{C}^2 : V^s_{31\bar{0}}(a,b) \cap V^u_{\bar{0}23}(a,b) \neq \emptyset \text{ tangentially}\}$$

and

$$\mathcal{T}^- \equiv \{(a,b) \in \mathbb{C}^2 : V^s_{41\bar{0}}(a,b) \cap V^u_{\overline{434124}}(a,b) \neq \emptyset \text{ tangentially}\}$$

and call them the *complex tangency loci*.

Complex Analytic Sets

A *complex analytic set* is the set of common zeros of finitely many analytic functions in $U \subset \mathbb{C}^n$. A general consideration yields that the complex tangency loci $\mathcal{T}^+$ and $\mathcal{T}^-$ form complex analytic sets, but possibly with singularities.

How to wipe out the singularities?

Lemma

Let $U_a, U_b \subset \mathbb{C}$ and let $\mathcal{T} \subset U_a \times U_b$ be a complex analytic set. If $\overline{\mathcal{T}} \cap (\partial U_a \times U_b) = \emptyset$, the projection $\pi_b : \mathcal{T} \to U_b$ is proper.

For $\mathcal{T}^\pm$, we can count the degree of π_b at $b = 0$; transversality of the quadratic family $p_a : x \mapsto x^2 - a$ at $a = 2$ yields that it is one. Proper of degree one $\Longrightarrow$ complex manifold (no singularity!).

Tin Can Argument

To verify the assumption of the previous lemma, we consider

$$\mathbb{C} \supset V \xrightarrow{\varphi} V^u_{\bar{0}}(a,b) \xrightarrow{f^2} V^u_{\bar{0}23}(a,b) \subset \mathcal{B}_3 \xrightarrow{\pi_u} U_3.$$

Theorem (Tin-Can)

When $b > 0$, the critical values of $\pi_u \circ f^2 \circ \varphi : V \to U_3$ are away from $\pi_u(V^s_{31\bar{0}}(a,b))$ for $(a,b) \in \partial_v \mathcal{F}$. Similar statement holds for the case $b < 0$ as well.

Rigorous Numerics

For all parameter values $(a,b) \in \partial_v \mathcal{F}$, we rigorously enclose the corresponding pieces of unstable and stable manifolds using set oriented method (GAIO-like algorithms) and check they do not intersect.

Last Message

Complex analytic methods are very powerful and convenient in the study of real dynamics of higher dimensions.